

THE LATTICE AND SIMPLEX STRUCTURE OF STATES ON PSEUDO EFFECT ALGEBRAS

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ABSTRACT. We study states, measures, and signed measures on pseudo effect algebras with some kind of the Riesz Decomposition Property, (RDP). We show that the set of all Jordan signed measures is always an Abelian Dedekind complete ℓ -group. Therefore, the state space of the pseudo effect algebra with (RDP) is either empty or a nonempty Choquet simplex or even a Bauer simplex. This will allow represent states on pseudo effect algebras by standard integrals.

1. INTRODUCTION

The seminal paper by Birkhoff and von Neumann [BiNe] showed that the events of quantum mechanical measurements do not fulfill the axioms of Boolean algebras and therefore also do not axioms of the classical probability theory presented by Kolmogorov [Kol]. It initiated the research of the mathematical foundations of quantum physics. Nowadays, there appeared a whole hierarchy of so-called quantum structures, like orthomodular lattices and posets, orthoalgebras, etc. Since the Nineties, we are intensively studying effect algebras that were introduced by Foulis and Bennett [FoBe]. An extensive source of information about effect algebras can be found in [DvPu]. Orthodox examples of the Hilbert space quantum mechanics are the system of closed subspaces, $\mathcal{L}(H)$, of a Hilbert space H (real, complex or quaternionic) and the system of all Hermitian operators, $\mathcal{E}(H)$, that are between the zero operator and the identity operator. An effect algebra E is a partial algebraic structure with a partially defined binary operation, $+$, that is commutative and it models join of “mutually exclusive” events. In many cases, it is an interval in a po-group (= partially ordered group), like $\mathcal{E}(H)$ is an interval in the po-group $\mathcal{B}(H)$ of all Hermitian operators on a Hilbert space H . A sufficient condition for an effect algebra to be an interval is e.g. the Riesz Decomposition Property (RDP, for short); and in such a case, E is an interval in a unique unital Abelian po-group (G, u) with interpolation, or equivalently, with (RDP), see [Rav] or [DvPu, Thm 1.7.17].

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In the last decade, there appeared many structures where the basic operation, $+$, is not necessarily commutative. The papers [DvVe1, DvVe2] present a non-commutative generalization of effect algebras, called *pseudo effect algebras*. In some important examples, they are also an interval in a unital po-group but not necessarily Abelian. Sufficient conditions for a pseudo effect algebra to be an interval in a unital po-group are stronger versions of (RDP), see [DvVe2] for more details.

Any measurement is accomplished by probabilistic reasoning. The quantum mechanical one is described by a state, an analogue of a probability measure. The state space of any pseudo effect algebra is an interesting structure that can be also void, see e.g. [Dvu1], but in general it is a convex compact Hausdorff topological space. In very important cases, it is a simplex and this allows then characterize states via an integral through a regular Borel probability measure, in some cases even in a unique way, see [Dvu3].

If an effect algebra satisfies (RDP), then it is an interval in an Abelian unital po-group with interpolation (RIP), so that it is a non-void simplex, [Dvu2, Thm 5.1]. If E is a pseudo effect with (RDP) that is an interval in a unital po-group, then it can happen that the state space is empty, [Dvu1]. In [Dvu3, Thm 4.2], we have showed that every interval pseudo effect algebra with (RDP) or an effect algebra with (RDP)₁ is a simplex.

The Riesz Decomposition Property is a weaker form of distributivity - it allows to make a joint refinement of two decompositions of the unit element. This is a reason why (RDP) fails to hold for $\mathcal{L}(H)$ and $\mathcal{E}(H)$.

We do not know whether every pseudo effect algebra with (RDP) is an interval in a unital po-group, this is known only for a stronger version (RDP)₁, [DvVe2, Thm 5.7]. Hence, we cannot directly apply the result from [Dvu3, Thm 4.2]. Therefore, we prove in the paper that the state space of a pseudo effect algebra with (RDP) is empty or a non-void Choquet simplex, Theorem 5.1. To prove that, we are studying the set of Jordan signed measures on a pseudo effect algebra with (RDP). We show that such a set is either a singleton containing only the zero measure or it is a non-trivial Abelian Dedekind complete ℓ -group (= lattice ordered). The simplex structure will be finally applied to represent a state as an integral through a unique regular Borel probability measure. We note that such a representation of states for MV-algebras (= effect algebras with (RDP)₂ = Phi-symmetric effect algebras, see [BeFo]) was proved in [Kro, Pan] and for effect algebras in [Dvu3].

The paper is organized as follows.

The elements of pseudo effect algebras are presented in Section 2. Section 3 describes the lattice structure of the group of all Jordan signed measures on a pseudo effect algebra with (RDP). Section 4 will describe some basic properties of Jordan signed measures that were known only for classical measures. Applications of the simplex structures of the state space, Choquet or Bauer simplices, for representation of states by integral are given in Section 5. The final concluding remarks are presented in Section 6.

2. PSEUDO EFFECT ALGEBRAS

Following [DvVe1, Dvu2], we say that a *pseudo effect algebra* is a partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, such that for all $a, b, c \in E$, the following holds

- (i) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;
- (ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;
- (iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;
- (iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then \leq is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. It is possible to show that $a \leq b$ if and only if $b = a + c = d + a$ for some $c, d \in E$. We write $c = a / b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a / b) = b,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a / 1$ for any $a \in E$.

For basic properties of pseudo effect algebras see [DvVe1, DvVe2]. We recall that if $+$ is commutative, E is said to be an *effect algebra*; for a comprehensive overview on effect algebras see e.g. [DvPu]. It is worthy to remark that effect algebras are equivalent to D-posets, where the basic operation is a difference of two comparable events, [KoCh].

We recall that a *po-group* (= partially ordered group) is a group G with a partial order, \leq , such that if $a \leq b$, $a, b \in G$, then $x + a + y \leq x + b + y$ for all $x, y \in G$. We denote by G^+ the set of all positive elements of G . If, in addition, \leq implies that G is a lattice, we call it an ℓ -group (= lattice ordered group). An element $u \in G^+$ is said to a *strong unit* if given $g \in G$, there is an integer $n \geq 1$ such that $g \leq nu$, and the couple (G, u) with a fixed strong unit is said to a *unital po-group*. The monographs like [Fuc, Gla] can serve as a basic source of information about partially ordered groups.

If (G, u) is a unital (not necessary Abelian) po-group with strong unit u , and

$$\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}, \quad (2.1)$$

then $(\Gamma(G, u); +, 0, u)$ is a pseudo effect algebra if we restrict the group addition $+$ to the set of all those $(x, y) \in \Gamma(G, u) \times \Gamma(G, u)$ that $x \leq u - y$.

Every pseudo effect algebra E that is isomorphic to some $\Gamma(G, u)$ is said to be an *interval pseudo effect algebra*.

According to [DvVe1], we introduce for pseudo effect algebras the following forms of the Riesz Decomposition Properties which in the case of commutative effect algebras can coincide:

- (a) For $a, b \in E$, we write a **com** b to mean that for all $a_1 \leq a$ and $b_1 \leq b$, a_1 and b_1 commute.
- (b) We say that E fulfils the *Riesz Interpolation Property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \leq c \leq b_1, b_2$.
- (c) We say that E fulfils the *weak Riesz Decomposition Property*, (RDP₀) for short, if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1$, $d_2 \leq b_2$ and $a = d_1 + d_2$.
- (d) We say that E fulfils the *Riesz Decomposition Property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$.
- (e) We say that E fulfils the *commutational Riesz Decomposition Property*, (RDP₁) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$

- there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \mathbf{com} d_3$.
- (f) We say that E fulfils the *strong Riesz Decomposition Property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

We have the implications

$$(\text{RDP}_2) \Rightarrow (\text{RDP}_1) \Rightarrow (\text{RDP}) \Rightarrow (\text{RDP}_0) \Rightarrow (\text{RIP}).$$

The converse of any of these implications does not hold. For commutative effect algebras we have

$$(\text{RDP}_2) \Rightarrow (\text{RDP}_1) \Leftrightarrow (\text{RDP}) \Leftrightarrow (\text{RDP}_0) \Rightarrow (\text{RIP}).$$

In addition, every pseudo effect algebra with $(\text{RDP})_2$ is an interval in a unital ℓ -group, [DvVe1, Prop 3.3].

In an analogous way we can define the same Riesz Decomposition Properties for a po-group G , where instead of E we deal with the positive cone G^+ .

We recall that an *MV-algebra* is an algebra $(A; \oplus, *, 0)$ of signature $\langle 2, 1, 0 \rangle$, where $(A; \oplus, 0)$ is a commutative monoid with neutral element 0, and for all $x, y \in A$

- (i) $(x^*)^* = x$,
- (ii) $x \oplus 1 = 1$, where $1 = 0^*$,
- (iii) $x \oplus (x \oplus y^*)^* = y \oplus (y \oplus x^*)^*$.

Sometimes it is used also a total binary operation \odot defined by $a \odot b := (a^* \oplus b^*)^*$.

If we define a partial addition, $+$, via $a + b$ is defined iff $a \leq b^*$, then $a + b = a \oplus b$, then $(A; +, 0, 1)$ is an effect algebra with $(\text{RDP})_2$, [DvVe2], or equivalently a Phi-symmetric effect algebra, [BeFo]; and it is an interval in an Abelian unital ℓ -group. Conversely, every lattice ordered effect algebra with (RDP) or equivalently, every effect algebra with $(\text{RDP})_2$ is in fact an MV-algebra.

3. SIGNED MEASURES AND JORDAN SIGNED MEASURES ON PSEUDO EFFECT ALGEBRAS

In the present section, we describe the lattice structures of the set of Jordan signed measures on a pseudo effect algebra satisfying (RDP) . We show that it is either trivial or a nontrivial Dedekind complete Riesz space.

Let E be a pseudo effect algebra. A *signed measure* on E is any mapping $m : E \rightarrow \mathbb{R}$ such that $m(a + b) = m(a) + m(b)$ whenever $a + b$ is defined in E . Then $m(0) = 0$ and $m(a^-) = m(a^\sim)$ for each $a \in E$. A *measure* is a positive signed measure m , i.e. $m(a) \geq 0$ for $a \in E$. Every measure is monotone on E . A *state* on E is any measure s such that $s(1) = 1$. Let $\mathcal{M}(E)$, $\mathcal{M}(E)^+$, and $\mathcal{S}(E)$ be the sets of all signed measures, measures, and states on E , respectively. It is clear that $\mathcal{M}(E) \neq \emptyset$ whilst $\mathcal{S}(E)$ can be empty. On $\mathcal{M}(E)$ we introduce a *weak topology* of signed measures defined as follows: a net of signed measures, $\{m_\alpha\}$, converges weakly to a signed measure m iff $\lim_\alpha m_\alpha(a) = m(a)$ for every $a \in E$. Then $\mathcal{M}(E)$ is a non-void compact Hausdorff topological space. Similarly, $\mathcal{S}(E)$ is a compact Hausdorff space that can be sometimes void. Moreover, $\mathcal{S}(E)$ is a convex set, i.e. if $s_1, s_2 \in \mathcal{S}(E)$ and $\lambda \in [0, 1]$, then $s = \lambda s_1 + (1 - \lambda)s_2 \in \mathcal{S}(E)$. A state s is *extremal* if from the property $s = \lambda s_1 + (1 - \lambda)s_2$ for some $s_1, s_2 \in \mathcal{S}(E)$ and $\lambda \in (0, 1)$, we conclude $s = s_1 = s_2$. Let $\partial_e \mathcal{S}(E)$ denote the set of all extremal states on E .

By the Krein–Mil’man Theorem, [Goo, Thm 5.17], every state on E is a weak limit of a net of convex combinations of extremal states. Hence, $\mathcal{S}(E) \neq \emptyset$ iff $\partial_e \mathcal{S}(E) \neq \emptyset$.

In what follows, we are inspired by the research in [Goo, pp. 37-41], where it was done for Abelian po-groups.

A mapping $d : E \rightarrow \mathbb{R}$ is said to be *subadditive* provided $d(0) = 0$ and $d(x+y) \leq d(x) + d(y)$ whenever $x+y \in E$.

Proposition 3.1. *Let E be a pseudo effect algebra with (RDP) and let $d : E \rightarrow \mathbb{R}$ be a subadditive mapping. For all $x \in E$, assume that the set*

$$D(x) := \{d(x_1) + \cdots + d(x_n) : x = x_1 + \cdots + x_n, x_1, \dots, x_n \in E, n \geq 1\} \quad (3.1)$$

is bounded above in \mathbb{R} . Then there is a signed measure $m : E \rightarrow \mathbb{R}$ such that $m(x) = \bigvee D(x)$ for all $x \in E$.

Proof. The map $m(x) := \bigvee D(x)$ is a well-defined mapping for all $x \in E$. It is clear that $m(0) = 0$ and now we are going to show that m is additive on E .

Let $x+y \in E$ be given. For all decompositions

$$x = x_1 + \cdots + x_n \text{ and } y = y_1 + \cdots + y_k$$

with all $x_i, y_j \in E$, we have $x+y = x_1 + \cdots + x_n + y_1 + \cdots + y_k$, that yields

$$\sum_i d(x_i) + \sum_j d(y_j) \leq m(x+y).$$

Therefore, $u+v \leq m(x+y)$ for all $u \in D(x)$ and $v \in D(y)$. Since \mathbb{R} is Dedekind complete, \bigvee is distributive with respect to $+$:

$$\begin{aligned} m(x) + m(y) &= \left(\bigvee D(x) \right) + m(y) = \bigvee_{u \in D(x)} (u + m(y)) \\ &= \bigvee_{u \in D(x)} \left(u + \left(\bigvee D(y) \right) \right) = \bigvee_{u \in D(x)} \bigvee_{v \in D(y)} (u + v) \\ &\leq m(x+y). \end{aligned}$$

Conversely, let $x+y = z_1 + \cdots + z_n$, where each $z_i \in E$. Then (RDP) implies that there are elements $x_1, \dots, x_n, y_1, \dots, y_n \in E$ such that $x = x_1 + \cdots + x_n$, $y = y_1 + \cdots + y_n$ and $z_i = x_i + y_i$ for $i = 1, \dots, n$. This yields

$$\sum_i d(z_i) \leq \sum_i (d(x_i) + d(y_i)) = \left(\sum_i d(x_i) \right) + \left(\sum_i d(y_i) \right) \leq m(x) + m(y),$$

and therefore, $m(x+y) \leq m(x) + m(y)$ and finally, $m(x+y) = m(x) + m(y)$ for all $x, y \in E$ such that $x+y$ is defined in E , so that m is a signed measure on E . \square

Let X be a poset. A mapping $m : X \rightarrow \mathbb{R}$ is said to be (i) *relatively bounded* provided that given any subset W of X which is bounded (above and below) in X , the set $m(W)$ is bounded in \mathbb{R} , (ii) *bounded* if $m(X)$ is bounded in \mathbb{R} .

We recall that if m is a signed measure on E , then m is relatively bounded iff m is bounded.

If G is a po-group, any group homomorphism $m : G \rightarrow \mathbb{R}$ is said to be a *signed measure* on G . Of course, if $m \neq 0$ is a measure that is relatively bounded on $G \neq \{0\}$, then it is not bounded on G .

Lemma 3.2. *If m is a signed measure on a unital po-group (G, u) , then m is relatively bounded iff m is bounded on the interval $[0, nu]$ for each $n \geq 1$. If, in addition, (G, u) satisfies (RDP), then m is relatively bounded iff m is bounded on the interval $[0, u]$.*

Proof. Indeed, one direction is clear, now suppose that m is bounded on each interval $[0, nu]$, and let W be bounded in G . Then $W \subseteq [a, b]$ and for some $a, b \in G$. There is an integer $n \geq 1$ such that $-a + b \leq nu$. Then $[a, b] = a + [0, -a + b] \subseteq a + [0, nu]$ and $m(W) \subseteq m(a) + m([0, nu]) \subseteq m(a) + [\alpha, \beta] = [m(a) + \alpha, m(a) + \beta]$ for some $\alpha, \beta \in \mathbb{R}$. This gives $m(W)$ is bounded in \mathbb{R} .

If, in addition, (G, u) satisfies (RDP), then $[0, nu] = [0, u] + \dots + [0, u]$. If $m([0, u])$ is bounded in \mathbb{R} , then $m([0, u]) \subseteq [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$. Then $m([0, nu]) \subseteq [n\alpha, n\beta]$. \square

Proposition 3.3. *Let E be a pseudo effect algebra with (RDP) and let $m : E \rightarrow \mathbb{R}$ be a signed measure. Then m is relatively bounded if and only if $m = m_1 - m_2$ for some measures m_1, m_2 on E .*

Proof. Assume that $m = m_1 - m_2$ for some two measures $m_1, m_2 \in \mathcal{M}(E)^+$. If $W \subseteq [a, b]$ in E , then $m_1(W) \subseteq [m_1(a), m_1(b)]$ and $m_2(W) \subseteq [m_2(a), m_2(b)]$. Then $m_1(a) - m_2(b) \leq m_1(b) - m_2(a)$ and $m(W) \subseteq [m_1(a) - m_2(b), m_1(b) - m_2(a)]$ that proves that m is relatively bounded.

Conversely, let m be relatively bounded. If we set $d(x) := m(x) \vee 0$ for all $x \in E$, then $d(0) = 0$. For all $x, y \in E$ such that $x + y$ is defined in E , we have

$$d(x + y) = (f(x) + f(y)) \vee 0 \leq (f(x) \vee 0) + (f(y) \vee 0) = d(x) + d(y),$$

so that d is subadditive.

Let us define $D(x)$ by (3.1) for each $x \in E$. We assert that $D(x)$ is bounded above in \mathbb{R} . By the assumption, there are elements $a, b \in H$ such that $f([0, x]) \subseteq [a, b]$. Fix a decomposition $x = x_1 + \dots + x_n$ with $x_i \in E$ for each $i = 1, \dots, n$. By [Goo, Lem 1.21], we have

$$\sum_{i=1}^n d(x_i) = \sum_{i=1}^n (m(x_i) \vee 0) = \left(\bigvee_{A \in 2^n} \left(\sum_{i \in A} m(x_i) \right) \right) \vee 0.$$

For all $A \in 2^n$, we have

$$0 \leq \sum_{i \in A} x_i \leq x, \text{ and } \sum_{i \in A} m(x_i) = m\left(\sum_{i \in A} x_i\right) \leq b.$$

Hence, $d(x_1) + \dots + d(x_n) \leq b \vee 0$, and consequently, $b \vee 0$ is an upper bound for $D(x)$ that proves the assertion.

By Proposition 3.1, there exists a signed measure m_1 on E such that $m_1(x) = \bigvee D(x)$ for all $x \in E$. Since $m_1(x) \geq d(x) \geq 0$, $m_1(x)$ is a measure, and $m_1(x) \geq d(x) \geq m(x)$ for all $x \in E$. Hence, $m_2 = m_1 - m$ is a measure on E , too. \square

A signed measure m on a pseudo effect algebra E is said to be *Jordan* if m can be expressed as a difference of two positive measures on E , and let $\mathcal{J}(E)$ be the set of all Jordan measures on E . It is clear that $\mathcal{J}(E)$ is nonempty because the zero mapping on E belongs to $\mathcal{J}(E)$.

For example, if $1 < \dim H < \infty$, then on $\mathcal{L}(H)$ there is a signed measure that is not Jordan, see e.g. [Dvu, 3.2.4], whilst if $\dim = \aleph_0$, then by the Dorofeev-Sherstnev Theorem, every σ -additive signed measure on $\mathcal{L}(H)$ is Jordan, [Dvu, Thm 3.2.20].

Proposition 3.3 says that a signed measure m on a pseudo effect algebra E with (RDP) is Jordan iff m is relatively bounded.

Given two signed measure $m_1, m_2 \in \mathcal{M}(E)$, we define $m_1 \leq^+ m_2$ whenever $m_2 - m_1$ is a positive measure. Then \leq^+ is a partial order on $\mathcal{M}(E)$ and $\mathcal{M}(E)$ is an Abelian po-group with respect to this partial order.

Let $(G; +, 0, \leq)$ be a po-group. A subgroup H of G is said to be *convex* if from $x \leq y \leq z$, where $x, z \in H$ and $y \in G$, we have $y \in H$. An *o-ideal* is any directed convex subgroup of G .

Proposition 3.4. *Let E be a pseudo effect algebra with (RDP), let $\mathcal{J}(E)$ be the set of all Jordan signed measures on E . Then $\mathcal{J}(E)$ is a nonempty o-ideal of the po-group $\mathcal{M}(E)$.*

Proof. Due to Proposition 3.3, $\mathcal{J}(E)$ equals the subgroup of $\mathcal{M}(E)$ generated by the positive measures. Therefore, $\mathcal{J}(E)$ is a directed subgroup of $\mathcal{M}(E)$.

Given $m_1 \in \mathcal{M}(E)$ and $m_2 \in \mathcal{J}(E)$ such that $0 \leq^+ m_1 \leq^+ m_2$, write $m_2 = m'_1 - m'_2$ for some measures $m'_1, m'_2 \in \mathcal{M}(E)^+$. Since $m_1 \leq^+ m_2 \leq^+ m'_1$, we have $m_1 = m'_1 - (m'_1 - m_1)$ with m'_1 and $m'_1 - m_1$ positive measures, and hence, $m_1 \in \mathcal{J}(E)$. This proves that $\mathcal{J}(E)$ is an o-ideal of $\mathcal{M}(E)$. \square

Theorem 3.5. *Let E be a pseudo effect algebra with (RDP).*

- (a) *The group $\mathcal{J}(E)$ of all Jordan signed measures on E is an Abelian Dedekind complete ℓ -group.*
- (b) *If $\{m_i\}_{i \in I}$ is a nonempty system of $\mathcal{J}(E)$ that is bounded above, and if $d(x) = \bigvee_i m_i(x)$ for all $x \in E$, then*

$$\left(\bigvee_i m_i \right) (x) = \bigvee \{d(x_1) + \cdots + d(x_n) : x = x_1 + \cdots + x_n, x_1, \dots, x_n \in E\}$$

for all $x \in E$.

- (c) *If $\{m_i\}_{i \in I}$ is a nonempty system of $\mathcal{J}(E)$ that is bounded below, and if $e(x) = \bigwedge_i f_i(x)$ for all $x \in E$, then*

$$\left(\bigwedge_i m_i \right) (x) = \bigwedge \{e(x_1) + \cdots + e(x_n) : x = x_1 + \cdots + x_n, x_1, \dots, x_n \in E\}$$

for all $x \in E$.

Proof. Let $t \in \mathcal{J}(E)$ be an upper bound for $\{m_i\}$. For any $x \in E$, we have $m_i(x) \leq t(x)$, so that the mapping $d(x) = \bigvee_i m_i(x)$ defined on E is a subadditive mapping. For any $x \in E$ and any decomposition $x = x_1 + \cdots + x_n$ with all $x_i \in E$, we conclude $d(x_1) + \cdots + d(x_n) \leq t(x_1) + \cdots + t(x_n) = t(x)$. Hence, $t(x)$ is an upper set for $D(x)$ defined by (3.1).

Proposition 3.1 entails there is a signed measure m on E such that $m(x) = \bigvee D(x)$. For every $x \in E$ and every m_i we have $m_i(x) \leq d(x) \leq m(x)$ that gives $m_i \leq^+ m$. The mappings $m - m_i$ are positive measures belonging to $\mathcal{M}(E)^+$ that gives $m \in \mathcal{J}(E)$. If $h \in \mathcal{J}(E)$ such that $m_i \leq^+ h$ for each $i \in I$, then $d(x) \leq h(x)$ for any $x \in E$. As above, we can show that $h(x)$ is also an upper bound for $D(x)$, whence $m(x) \leq h(x)$ for any $x \in E$ that gives $m \leq^+ h$. In other words, we have proved that m is the supremum of $\{m_i\}_{i \in I}$, and its form is given by (b).

Applying the order anti-automorphism $z \mapsto -z$ in \mathbb{R} , we see that infima exist in $\mathcal{J}(E)$ for any bounded below system $\{m_i\}_{i \in I}$, and their form is given by (c).

By Proposition 3.3, $\mathcal{J}(E)$ is directed, combining (b) and (c), we see that $\mathcal{J}(E)$ is an Abelian Dedekind complete ℓ -group. \square

For finite joins and meets of Jordan signed measures, Theorem 3.5 can be reformulated as follows.

Theorem 3.6. *If E is a pseudo effect algebra with (RDP), then the group $\mathcal{J}(E)$ of all Jordan signed measures on E is an Abelian Dedekind complete lattice ordered real vector space. Given $m_1, \dots, m_n \in \mathcal{J}(E)$,*

$$\left(\bigvee_{i=1}^n m_i \right) (x) = \sup \{ m_1(x_1) + \dots + m_n(x_n) : x = x_1 + \dots + x_n, x_1, \dots, x_n \in E \},$$

$$\left(\bigwedge_{i=1}^n m_i \right) (x) = \inf \{ m_1(x_1) + \dots + m_n(x_n) : x = x_1 + \dots + x_n, x_1, \dots, x_n \in E \},$$

for all $x \in E$.

Proof. Due to Theorem 3.5, $\mathcal{J}(E)$ is an Abelian Dedekind complete ℓ -group. It is evident that it is a Riesz space, i.e., a lattice ordered real vector space.

Take $m_1, \dots, m_n \in \mathcal{J}(E)$ and let $m = m_1 \vee \dots \vee m_n$. For any $x \in E$ and $x = x_1 + \dots + x_n$ with $x_1, \dots, x_n \in E$, we have $m_1(x_1) + \dots + m_n(x_n) \leq m(x_1) + \dots + m(x_n) = m(x)$. Due to Theorem 3.5, given an arbitrary real number $\epsilon > 0$, there is a decomposition $x = y_1 + \dots + y_k$ with $y_1, \dots, y_k \in E$ such that

$$\sum_{j=1}^k \max \{ m_1(y_j), \dots, m_n(y_j) \} > m(x) - \epsilon.$$

If $k < n$, we can add the zero elements to the decomposition, if necessary, so that without loss of generality, we can assume that $k \geq n$.

We note that if $a, b \in E$ are given such that $a + b$ is defined in E , the elements $a', a'' \in E$ such that $a + b = b + a'$ and $b + a = a'' + b$ are said to be (right and left) conjugates of a by b . Since \mathbb{R} is Abelian, for any $h \in \mathcal{J}(E)$, $h(a') = h(a) = h(a'')$.

We decompose the set $\{1, \dots, k\}$ into mutually disjoint sets $J(1), \dots, J(n)$ such that

$$J(i) := \{j \in \{1, \dots, k\} : \max \{ m_1(y_j), \dots, m_n(y_j) \} = m_i(y_j) \}.$$

Assume $J(1) = \{j_{t_1}, \dots, j_{n_1}\}$. Then the element $x_1 := x_{j_{t_1}} + \dots + x_{j_{n_1}}$ is defined in E .

The element x can be expressed in the form $x = x_{j_{t_1}} + \dots + x_{j_{n_1}} + x'_j + \dots + x'_k$, where $x'_j, \dots, x'_k \in E$ are conjugates of x_j, \dots, x_k .

In a similar way, let $J(2) = \{j_{t_2}, \dots, j_{n_2}\}$ and let $x_2 = y_{j_{t_2}} + \dots + y_{j_{n_2}}$. Again, we can express x in the form $x = x_1 + x_2 + y''_s + \dots + y''_k$, where y''_t 's are appropriate conjugates of y'_s, \dots, y'_k . Processing in this way for each $J(i) = \{j_{t_i}, \dots, j_{n_i}\}$, we define the element $x_i = c_{t_{j_{t_i}}} + \dots + c_{t_{j_{n_i}}}$, where $c_{t_{j_s}}$ is an appropriate conjugate of the element $y_{t_{j_s}}$. Then $x = x_1 + \dots + x_n$, and

$$\sum_{i=1}^n m_i(x_i) = \sum_{i=1}^n \sum_{j \in J(i)} m_i(y_j) = \sum_{i=1}^k \max \{ m_1(y_j), \dots, m_n(y_j) \} > m(x) - \epsilon.$$

This implies $m(x)$ equals the given supremum.

The formula for $(m_1 \wedge \cdots \wedge m_n)(x)$ can be obtained applying the order anti-automorphism $z \mapsto -z$ holding in \mathbb{R} . \square

4. JORDAN SIGNED MEASURES

Using the results of the previous Section, we will show some interesting properties of signed measures, like a Jordan decomposition, variation, etc.

Let E be a pseudo effect algebra with (RDP), and let $0 : E \rightarrow \{0\}$ be the zero signed measure. Then $\mathcal{J}(E)$ is a nontrivial Abelian ℓ -group, i.e., $\mathcal{J}(E) \supset \{0\}$ iff E admits at least one state. Moreover, 0 is the zero element of the ℓ -group $\mathcal{J}(E)$. We recall that if E is an effect algebra with (RDP), then $\mathcal{S}(E)$ is always nonempty.

We say that a convex subset F of a convex set K is a *face* if $x = \lambda x_1 + (1 - \lambda)x_2 \in F$, $0 < \lambda < 1$, entail $x_1, x_2 \in F$. For example, if x is an extreme point of K , then the singleton $\{x\}$ is a face, and for any $X \subseteq K$, there is the face generated by X . Due to [Goo, Prop 5.7], the face F generated by X is the set of those points $x \in K$ for which there exists a positive convex combination $\lambda x + (1 - \lambda)y = z$ with $y \in K$ and z belonging to the convex hull of X .

In particular, the face of K generated by a point $z \in K$ consists precisely of those points $x \in K$ for which there exists a positive convex combination $\lambda x + \beta y = z$ with $y \in K$.

Lemma 4.1. *Let E be a pseudo effect algebra and let X be a subset of $\mathcal{S}(E)$. Then a state $s \in \mathcal{S}(E)$ belongs to the face generated by X if and only if $s \leq^+ \alpha t$ for some positive constant α and some state t in the convex hull of X .*

Proof. If a state s belongs to the face generated by X , by the note just before this lemma, there exists a positive convex combination $\lambda s + (1 - \lambda)s' = t$, where $s' \in \mathcal{S}(E)$ and t belongs to the convex hull of X . Then $\lambda s \leq^+ t$ so that $s \leq^+ t/\lambda$.

Conversely, if $s \leq^+ \alpha t$ for some $\alpha > 0$ and some state t in the convex hull of X . Then $\alpha t - s$ is a measure, so that $\alpha t - s = \beta s'$ for some $\beta \geq 0$. Now $s + \beta s' = \alpha t$ and $1 + \beta = s(1) + \beta s'(1) = \alpha t(1) = \alpha$ that yields $1/\alpha + \beta/\alpha = 1$. This gives $0 \leq \lambda := 1/\alpha \leq 1$ and $\lambda s + (1 - \lambda)s' = t$. Since t belongs to the face generated by X , so does s . \square

Now we show that if s_1 and s_2 are two states on E , then $s_1 \wedge s_2$ and $s_1 \vee s_2$ are not necessarily states.

Proposition 4.2. *Let E be a pseudo effect algebra with (RDP). Let F_1 and F_2 be the faces generated by states s_1 and s_2 , respectively, on E . The following statements are equivalent:*

- (i) $F_1 \cap F_2 = \emptyset$.
- (ii) $s_1 \wedge s_2 = 0$.
- (iii) $s_1 \vee s_2 = s_1 + s_2$.
- (iv) *Given $x \in E$ and any $\epsilon > 0$, there exists $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $s(x \setminus x_i) < \epsilon$.*

In particular, if s_1 and s_2 are two distinct extremal state on E , then $s_1 \wedge s_2 = 0$.

Proof. (i) \Rightarrow (ii). Assume that s_1 and s_2 belongs to mutually disjoint faces of $\mathcal{S}(E)$. If $s_1 \wedge s_2 > 0$, there is a state s and a real number $\alpha > 0$ such that $s_1 \wedge s_2 = \alpha s$. Then $\alpha s \leq^+ s_1$ and $\alpha s \leq^+ s_2$ and $s \leq^+ s_1/\alpha$ and $s \leq^+ s_2/\alpha$. Lemma 4.1 implies s

belongs to the face generated by s_1 and as well to the one generated by s_1 that is absurd, so that $s_1 \wedge s_2 = 0$.

(ii) \Rightarrow (i). Let F_1 and F_2 be the faces generated by s_1 and s_2 , respectively. We state that $F_1 \cap F_2 = \emptyset$. If not, there is a state $s \in F_1 \cap F_2$ and by Lemma 4.1, $s \leq^+ \alpha_1 s_1$ and $s \leq^+ \alpha_2 s_2$. If $\alpha = \max\{\alpha_1, \alpha_2\}$, then $s \leq^+ \alpha s_1$ and $s \leq^+ \alpha s_2$ and $s/\alpha \leq^+ s_1, s_2$ and therefore, $s/\alpha \leq^+ s_1 \wedge s_2 = 0$ that gives a contradiction.

(i) \Leftrightarrow (iii). It follows from the basic properties of ℓ -groups, see e.g. [Fuc, C p.67], $(s_1 \vee s_2) + (s_1 \wedge s_2) = s_1 + s_2$.

(ii) \Rightarrow (iv). By Theorem 3.6, given x and $\epsilon > 0$, there are $x'_1, x'_2 \in E$ such that $x = x'_2 + x'_1$ and $s_1(x'_2) + s_2(x'_1) < \epsilon$. Then $x'_2 = x \setminus x'_1$ and $x'_1 = x'_2 / x$, and $s_1(x'_2) = s_1(x \setminus x'_1) < \epsilon$ and $s_2(x'_1) = s_1(x'_2 / x) = -s_1(x'_2) + s_1(x) = s_1(x) - s_1(x'_2) = s_1(x \setminus x'_2) < \epsilon$.

But $x = x'_1 + x''_2$, where x''_2 is a conjugate of x'_2 by x'_1 . Then $s_2(x''_2) = s_2(x'_2)$. If we set $x_1 = x'_1$ and $x_2 = x''_2$, we have $s_1(x \setminus x_1) = s_1(x \setminus x'_1) < \epsilon$, $s_2(x \setminus x_2) = s_2(x) - s_2(x_2) = s_2(x) - s_2(x''_2) = s_2(x) - s_2(x'_2) = s_2(x \setminus x'_2) < \epsilon$ and $x = x_1 + x_2$.

(iv) \Rightarrow (ii). Given $\epsilon > 0$ and $x \in E$, there is a decomposition $x = x_1 + x_2$ such that $s_i(x \setminus x_i) < \epsilon/2$ for $i = 1, 2$. Hence, $x = x_2 + x'_1$, where x'_1 is a conjugate of x_1 . If we set $y = x_2$ and $z = x'_1$, then using $x_1 = x \setminus x_2$ and $x_2 = x_1 / x$, we have $s_1(y) = s_1(x_2) = s_1(x_1 / x) = s_1(x \setminus x_1) < \epsilon/2$ and $s_2(z) = s_2(x'_1) = s_2(x \setminus x_2) < \epsilon/2$, so that $s(y) + s(z) < \epsilon$. By Theorem 3.6, this means $s_1 \wedge s_2 = 0$.

Finally, if s_1 and s_2 are two distinct extremal states, then the singletons $\{s_1\}$ and $\{s_2\}$ are mutually disjoint faces. Hence, $s_1 \wedge s_2 = 0$. \square

Proposition 4.3. *Let s_1, s_2 be two states on a pseudo effect algebra E with (RDP). Then $s_1 \wedge s_2 \in \mathcal{S}(E)$ if and only if $s_1 = s_2$ and if and only if $s_1 \vee s_2$ is a state.*

Given $\lambda \in [0, 1]$, let $s_\lambda := \lambda s_1 + (1 - \lambda)s_2 \in \mathcal{S}(E)$. Then $s_1 \wedge s_2 = \bigwedge \{s_\lambda : \lambda \in [0, 1]\} \in \mathcal{M}^+(E)$.

Proof. Let $s = s_1 \wedge s_2 \in \mathcal{S}(E)$. Then $s \leq^+ s_1$ and $s \leq^+ s_2$. Therefore, $s_i - s$ is a positive measure. Since $s_i(1) - s(1) = 0$ for $i = 1, 2$, we see that $s_1 = s = s_2$. The converse statement is evident. The second equivalency follows from the ℓ -group equality $(s_1 \wedge s_2) + (s_1 \vee s_2) = s_1 + s_2$.

Let $s = s_1 \wedge s_2 \in \mathcal{M}^+(E)$. Given $\lambda \in [0, 1]$, we have $\lambda s \leq^+ \lambda s_1$ and $(1 - \lambda)s \leq^+ (1 - \lambda)s_2$ so that $s = \lambda s + (1 - \lambda)s \leq^+ \lambda s_1 + (1 - \lambda)s_2$. Hence $s \leq^+ s_0 := \bigwedge \{s_\lambda : \lambda \in [0, 1]\}$. If we set $\lambda = 1$ or $\lambda = 0$, we see that $s_1, s_2 \in \{s_\lambda : \lambda \in [0, 1]\}$. Therefore, $s_0 \leq^+ s$. \square

A signed measure m on a pseudo effect algebra E is σ -additive if, $\{a_n\} \nearrow a$, i.e. $a_n \leq a_{n+1}$ for each $n \geq 1$ and $\bigvee_n a_n = a$, then $m(a) = \lim_n m(a_n)$. A measure m is σ -additive iff $a_n \searrow 0$ entails $m(a_n) \searrow 0$.

Proposition 4.4. *If m_1 and m_2 are σ -additive measures on a pseudo effect algebra with (RDP), so are $m_1 \vee m_2$ and $m_1 \wedge m_2$.*

Proof. Let $a_n \searrow 0$. Due to Theorem 3.6, $m_1(a_n) + m_2(a_n) \geq (m_1 \vee m_2)(a_n) \geq 0$ so that $(m_1 \vee m_2)(a_n) \searrow 0$. Similarly, $m_1(a_n) \geq (m_1 \wedge m_2)(a_n) \geq 0$ and $(m_1 \wedge m_2)(a_n) \searrow 0$. \square

Theorem 3.6 allows us to define, for any Jordan signed measure m , its positive and negative parts, m^+ and m^- , via

$$m^+ := m \vee 0 \quad \text{and} \quad m^- := -(m \wedge 0).$$

Then $m = m^+ - m^-$, $(-m)^+ = m^-$, and $(-m)^- = m^+$.

Theorem 3.6 says that

$$m^+(a) = \sup\{m(x) : x \leq a\} \quad \text{and} \quad m^-(a) = \inf\{m(x) : x \leq a\} \quad (4.1)$$

for each $a \in E$.

The decomposition $m = m^+ - m^-$ is said to be *Jordan*, and if $m = m_1 - m_2$ for some positive measures m_1, m_2 on E , then $m^+ \leq^+ m_1$ and $m^- \leq^+ m_2$. Moreover, we define an *absolute value*, $|m|$, of m via

$$|m| = m^+ + m^-.$$

Therefore, if $\mathcal{S}(E) \neq \emptyset$, every Jordan signed measure m can be uniquely expressed in the form

$$m = \alpha_1 s_1 - \alpha_2 s_2, \quad (4.2)$$

where α_1, α_2 are real numbers and s_1, s_2 are states such that $\alpha_1 s_1 = m^+$ and $\alpha_2 s_2 = m^-$, we call it a *canonical Jordan decomposition* of m .

The measures m^+, m^- and $|m|$ are sometimes called also an *upper* or *positive variation*, a *lower* or *negative variation* and a *total variation* of m , respectively.

Proposition 4.5. *For any Jordan signed measure m on a pseudo effect algebra E with (RDP), we define a mapping $v_m : E \rightarrow \mathbb{R}$ by*

$$v_m(x) := \sup\{|m(x_1)| + \cdots + |m(x_n)| : x = x_1 + \cdots + x_n, n \geq 1\}. \quad (4.3)$$

Then $v_m = |m|$.

Proof. Let $x = x_1 + \cdots + x_n$. Then $|m(x_1)| + \cdots + |m(x_n)| \leq |m|(x_1) + \cdots + |m|(x_n) = |m|(x)$, so that $v_m(x) \leq |m|(x)$. Due to (4.1), $m^+(x), m^-(x) \leq v_m(x)$ for each $x \in E$. We assert that v_m is subadditive, i.e., $v_m(x + y) \leq v_m(x) + v_m(y)$ whenever $x + y \in E$. Indeed, if $x + y = z_1 + \cdots + z_n$, (RDP) entails that there are $x_1, \dots, z_n, y_1, \dots, y_n \in E$ such that $x = x_1 + \cdots + z_n$ and $y = z_1 + \cdots + z_n$. Then $|m(z_1)| + \cdots + |m(z_n)| \leq \sum_i |m(x_i)| + \sum_i |m(y_i)| \leq v_m(x) + v_m(y)$, so that $v_m(x + y) \leq v_m(x) + v_m(y)$.

According to (3.1), we define the set

$$V_m(x) = \{v_m(x_1) + \cdots + v_m(x_n) : x = x_1 + \cdots + x_n\}.$$

This set is bounded in \mathbb{R} , its upper bound is $|m|(x)$. Proposition 3.1 yields that the functional $V(x) = \sup V_m(x)$, $x \in E$, is a positive measure on E . It is clear that $v_m(x) \leq V(x) \leq |m|(x)$ for each $x \in E$.

We show that $v_m = V$. Given $\epsilon > 0$, there is a decomposition $x = x_1 + \cdots + x_n$ such that $\sum_i v_m(x_i) > V(x) - \epsilon$. For any $i = 1, \dots, n$, there is a finite decomposition of each $x_i = \sum_j x_i^j$ such that $\sum_j |m(x_i^j)| \geq v_m(x_i) - \epsilon/n$. Therefore,

$$\sum_{i=1}^n \sum_j |m(x_i^j)| > \sum_{i=1}^n (v_m(x_i) - \epsilon/n) = \sum_{i=1}^n v_m(x_i) - \epsilon > V(x) - 2\epsilon.$$

This entails, $v_m(x) \geq V(x) - 2\epsilon$. Since ϵ was arbitrary, $v_m(x) \geq V(x)$, consequently, $v_m(x) = V(x)$ for any $x \in E$.

Since $m^+ \wedge m^- = 0$, the ℓ -group properties imply $|m| = m^+ + m^- = m^+ \vee m^-$. Since $m^+, m^+ \leq^+ v_m = V \leq^+ |m|$, we have $v_m = |m|$. \square

Let $\{a_n\}$ be a sequence of elements of a pseudo effect algebra E such that $b_n = a_1 + \dots + a_n$ exists for each $n \geq 1$. If $a = \bigvee_n b_n$ is defined in E , we write $a := a_1 + a_2 + \dots = \sum_n a_n$. Let $\{a_n\} \nearrow a$. If we set $a'_1 = a_1$ and $a'_n = a_{n-1} / a_n$ for each $n \geq 2$, then $a'_1 + \dots + a'_n = a_n$ for each $n \geq 1$ and $a = \sum_n a'_n$. Hence, a signed measure m is σ -additive iff $a = \sum_n a_n$ entails $m(a) = \sum_n m(a_n)$.

A pseudo effect algebra is said to be *monotone σ -complete* if $a_1 \leq a_2 \leq a_n \leq \dots$, then $a = \bigvee_n a_n$ is defined in E . We say that E satisfies σ -(RDP) if $a_1 + a_2 = b_1 + b_2 + \dots$, then there are two sequences $\{c_{1n}\}_n$ and $\{c_{2n}\}$ such that $a_i = \sum_n c_{in}$ for $i = 1, 2$ and $b_n = c_{1n} + c_{2n}$ for each $n \geq 1$.

Proposition 4.6. *Let E be a pseudo effect algebra with σ -(RDP). If m is a σ -additive Jordan signed measure, so is m^+, m^- and $|m|$.*

Proof. Assume $\{a_n\} \nearrow a$. If we set $a'_1 = a_1$ and $a'_n = a_{n-1} / a_n$ for each $n \geq 2$, then $a'_1 + \dots + a'_n = a_n$ for each $n \geq 1$ and $a = \sum_n a'_n$. Then $m(a) = \sum_n m(a'_n)$.

We show that $|m| = v_m$ is σ -additive. We have $v_m(a) \geq v_m(a_n)$ so that $v_m(a) \geq \lim v_m(a_n)$. Now assume $a = x_1 + \dots + x_k$. The σ -(RDP) entails that there is k many sequences $\{c_{jn}\}_n$ for $j = 1, \dots, k$ such that $x_j = \sum_n c_{jn}$ and $a'_n = \sum_{j=1}^k c_{jn}$ for each $n \geq 1$. Check

$$\begin{aligned} \sum_{j=1}^k |m(x_j)| &= \sum_{j=1}^k |m(\sum_n c_{jn})| = \sum_{j=1}^k |\sum_n m(c_{jn})| \\ &\leq \sum_n \sum_{j=1}^k |m(c_{jn})| \leq \sum_n v_m(a_n), \end{aligned}$$

so that $v_m(a) \leq \sum_n v_m(a_n) \leq \lim_n v_m(a_n)$ and $v_m = |m|$ is σ -additive. Because $m + 2m^+ = |m|$, we see that m^+ is σ -additive, consequently, so is m^- . \square

Suppose that E admits at least one state. Given a positive measure m on E with $m(1) > 0$, let $\mathcal{J}(m) = [0, m] := \{t \in \mathcal{J}(E) : 0 \leq^+ t \leq^+ m\}$ be an interval in $\mathcal{J}(E)$. We can define on it an MV-structure by $s \oplus t := (s + t) \wedge m$, $s \odot t := \{s + t - m\} \vee 0$, and $s^* = m - s$ for all $s, t \in \mathcal{J}(m)$. Then $(\mathcal{J}(m); \oplus, *, 0)$ is an MV-algebra, where $m = 0^*$ is the unit element of $\mathcal{J}(m)$.

The partial operation, $+$, on $\mathcal{J}(m)$, is defined as follows: $s + t$ is defined in $\mathcal{J}(m)$ iff $s + t \leq^+ m$, or equivalently, it coincides with the restriction of the standard addition of the functions s and t belongs to $\mathcal{J}(m)$.

It is clear that the state space of $\mathcal{J}(m)$ is non-void. Let $a \in E$ be a fixed element. The mapping $\mu_a : \mathcal{J}(m) \rightarrow [0, 1]$ defined by $\mu_a(s) := s(a)$, $s \in \mathcal{J}(m)$, is a state on $\mathcal{J}(E)$.

Moreover, the system of states $\{\mu_a : a \in E\}$ is *order-determining*, i.e. $\mu_a(s) \leq \mu_a(t)$ for all $a \in E$, implies $s \leq^+ t$.

5. SIMPLEX STRUCTURE OF PSEUDO EFFECT ALGEBRAS AND INTEGRALS

This is the main section of the paper. We show that if a pseudo effect algebra satisfies (RDP), then its state space is either empty or a non-empty simplex. This will allow represent states by standard integrals.

The following notions on convex sets can be found e.g. in [Goo]. Let K_1, K_2 be two convex sets. A mapping $f : K_1 \rightarrow K_2$ is said to be *affine* if it preserves all

convex combinations, and if f is also injective and surjective such that also f^{-1} is affine, f is said to be an *affine isomorphism* and K_1 and K_2 are *affinely isomorphic*.

We recall that a *convex cone* in a real linear space V is any subset C of V such that (i) $0 \in C$, (ii) if $x_1, x_2 \in C$, then $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^+$. A *strict cone* is any convex cone C such that $C \cap -C = \{0\}$, where $-C = \{-x : x \in C\}$. A *base* for a convex cone C is any convex subset K of C such that every non-zero element $y \in C$ may be uniquely expressed in the form $y = \alpha x$ for some $\alpha \in \mathbb{R}^+$ and some $x \in K$.

We recall that in view of [Goo, Prop 10.2], if K is a non-void convex subset of V , and if we set

$$C = \{\alpha x : \alpha \in \mathbb{R}^+, x \in K\},$$

then C is a convex cone in V , and K is a base for C iff there is a linear functional f on V such that $f(K) = 1$ iff K is contained in a hyperplane in V which misses the origin.

Any strict cone C of V defines a partial order \leq_C via $x \leq_C y$ iff $y - x \in C$. It is clear that $C = \{x \in V : 0 \leq_C x\}$. A *lattice cone* is any strict convex cone C in V such that C is a lattice under \leq_C .

A *simplex* in a linear space V is any convex subset K of V that is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex K in a locally convex Hausdorff space is said to be (i) *Choquet* if K is compact, and (ii) *Bauer* if K and $\partial_e K$ are compact, where $\partial_e K$ is the set of extreme points of K .

A simplex is a generalization of a classical simplex in \mathbb{R}^n , and we recall that no disc or no convex quadrilateral in the plane are not simplices.

Theorem 5.1. *If E is a pseudo effect algebra with (RDP), then either $\mathcal{S}(E)$ is empty or it is a nonempty Choquet simplex.*

Proof. Assume that $\mathcal{S}(E)$ is nonempty. Then the positive cone $\mathcal{M}(E)^+ = \mathcal{J}(E)^+$ of the Abelian Dedekind complete ℓ -group $\mathcal{J}(E)$ consists of all positive measures on E , so that $\mathcal{J}(E)^+ = \{\alpha s : \alpha \in \mathbb{R}^+, s \in \mathcal{S}(E)\}$. Since $\mathcal{S}(E)$ lies in the hyperplane $\{m \in \mathcal{J}(E) : m(u) = 1\}$ which misses the origin, $\mathcal{S}(E)$ is a base for $\mathcal{J}(E)^+$, and $\mathcal{S}(E)$ is a simplex. On the other hand, $\mathcal{S}(E)$ is compact, so that $\mathcal{S}(E)$ is a Choquet simplex. \square

We note that if E is an effect algebra with (RDP) and $0 \neq 1$, then E admits at least one state because then $E = \Gamma(G, u)$ for some unital Abelian interpolation po-group (G, u) ; now it is enough to apply [Goo, Cor 4.4]. Hence, its state space is always a non-empty Choquet simplex. If an effect algebra E does not satisfy (RDP), then its state space is not necessarily a simplex; for instance, this is the case for $E = \mathcal{E}(H)$, $\dim H > 2$. On the other hand, the state space of a commutative C^* -algebra or the trace space of a general C^* are simplices, [AlSc, Thm 4.4, p. 7] or [BrRo, Ex 4.2.6].

On the other hand, it is important to recall that according to a delicate result of Choquet [Alf, Thm I.5.13], for any pseudo effect algebra E , $\partial_e \mathcal{S}(E)$ is always a Baire space in the relativized topology induced by the topology of $\mathcal{S}(E)$, i.e. the Baire Category Theorem holds for $\partial_e \mathcal{S}(E)$.

Remark 5.2. *Theorem 5.1 was proved for a pseudo effect algebra that is an interval pseudo effect algebra, i.e., $E = \Gamma(G, u)$, for a unital po-group (G, u) with (RDP).*

However, we do not know whether every pseudo effect algebra with (RDP) is an interval pseudo effect $\Gamma(G, u)$, where also (G, u) satisfies (RDP), it was necessary to prove Theorem 5.1 in full details.

If a pseudo effect algebra E satisfies $(\text{RDP})_2$, then according to [Dvu3, Thm 4.4], the state space of E is either the empty set or a nonempty Bauer simplex.

Example 5.3. *There is a pseudo effect algebra E with (RDP) but $(\text{RDP})_2$ fails to hold in E such that $\mathcal{S}(E)$ is a non-void Bauer simplex.*

Proof. Let \mathbb{Q} be the set of all rational numbers and let $G = \mathbb{Q} \times \mathbb{Q}$ be ordered by the strict ordering, i.e. $(g_1, g_2) \leq (h_1, h_2)$ iff $g_1 < h_1$ and $g_2 < h_2$ or $g_1 = h_1$ and $g_2 = h_2$. If we set $u = (1, 1)$, then $E = \Gamma(G, u)$ is an effect algebra with (RDP) that is not a lattice. If $s_0(g, h) := h$ and $s_1(g, h) := g$, then s_0 and s_1 are unique extremal states on E , and every state s is of the form $s = s_\lambda := \lambda s_1 + (1 - \lambda)s_0$, $\lambda \in [0, 1]$, for more details, see [BCD, Ex 4.2]. \square

A pseudo effect algebra E has the *Bauer simplex property* ((BSP) for short), if $\mathcal{S}(E)$ is a non-void Bauer simplex.

Let K be a compact convex subset of a locally convex Hausdorff space. A mapping $f : K \rightarrow \mathbb{R}$ is said to be *affine* if, for all $x, y \in K$ and any $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) = \lambda f(x) + (1 - \lambda)f(y)$. Let $\text{Aff}(K)$ be the set of all continuous affine functions on K . Then $\text{Aff}(K)$ is a unital po-group with the strong unit 1 which is a subgroup of the po-group $C(K)$ of all continuous real-valued functions on K (we recall that, for $f, g \in C(K)$, $f \leq g$ iff $f(x) \leq g(x)$ for any $x \in K$). In addition, $C(K)$ is an ℓ -group and the function 1 is its strong unit.

We note that if E is a pseudo effect algebra such that $\mathcal{S}(E) \neq \emptyset$, given $a \in E$, let $\hat{a} : \mathcal{S}(E) \rightarrow [0, 1]$ such that $\hat{a}(s) := s(a)$, $s \in \mathcal{S}(E)$. Then $\hat{a} \in \text{Aff}(\mathcal{S}(E))$.

If K is a compact Hausdorff topological space, let $\mathcal{B}(K)$ be the Borel algebra of K generated by all open subsets of K . Let $\mathcal{M}_1^+(K)$ denote the set of all probability measures, that is, all positive regular σ -additive Borel measures μ on $\mathcal{B}(K)$. We recall that a Borel measure μ is called *regular* if each value $\mu(Y)$ can be approximated by closed subspaces of Y as well by open subsets O such that $Y \subseteq O$.

We recall that if $x \in K$, then the Dirac measure δ_x defined by $\delta(A) := \chi_A(x)$, $A \in \mathcal{B}(K)$, is a regular Borel probability measure.

For two measures μ and ν we define the Choquet equivalence \sim defined by

$$\mu \sim \lambda \quad \text{iff} \quad \int_K f d\mu = \int_K f d\lambda, \quad f \in \text{Aff}(K).$$

If μ and λ are nonnegative regular Borel measures on a convex compact set K , we introduce for them the *Choquet ordering* defined by

$$\mu \prec \lambda \quad \text{iff} \quad \int_K f d\mu \leq \int_K f d\lambda, \quad f \in \text{Con}(K),$$

where $\text{Con}(K)$ is the set of all continuous convex functions f on K (that is $f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2)$ for $x_1, x_2 \in K$ and $\alpha \in [0, 1]$). Then \prec is a partial order on the cone of nonnegative measures. The fact $\lambda \prec \mu$ and $\mu \prec \lambda$ implies $\lambda = \mu$ follows from the fact that $\text{Con}(K) - \text{Con}(K)$ is dense in $C(K)$.

Moreover, for any probability measure λ there is a maximal probability measure μ in Choquet's ordering such that $\mu \succ \lambda$, [Phe, Lem 4.1].

We recall that the Choquet ordering $\mu \prec \nu$ between two probability measures μ and ν roughly speaking means that ν is located further out than μ towards the set of extremal points where the convex function have large values, [AlSc, p. 8].

Theorem 5.4. *Let E be a pseudo effect algebra with (RDP) and with $\mathcal{S}(E) \neq \emptyset$. Let s be a state on E . Let $\psi : E \rightarrow \text{Aff}(\mathcal{S}(E))$ be defined by $\psi(a) := \hat{a}$, $a \in E$, where \hat{a} is a mapping from $\mathcal{S}(E)$ into $[0, 1]$ such that $\hat{a}(s) := s(a)$, $s \in \mathcal{S}(E)$. Then there is a unique state \tilde{s} on the unital po-group $(\text{Aff}(\mathcal{S}(E)), 1)$ such that $\tilde{s}(\hat{a}) = s(a)$ for any $a \in E$.*

The mapping $s \mapsto \tilde{s}$ defines an affine homeomorphism from the state space $\mathcal{S}(E)$ onto $\mathcal{S}(\Gamma(\text{Aff}(\mathcal{S}(E)), 1))$.

Proof. Since E is a pseudo effect algebra such that $\mathcal{S}(E)$ is non-void, Theorem 5.1 asserts that $\mathcal{S}(E)$ is a nonempty Choquet simplex. We define $\hat{E} := \{\hat{a} : a \in E\} \subset \text{Aff}(\mathcal{S}(E))$ and let $\text{Aff}(E)$ be the Abelian subgroup of $\text{Aff}(\mathcal{S}(E))$ generated by \hat{E} . Given $s \in \mathcal{S}(E)$, let \tilde{s} be a mapping defined on the unital Abelian po-group with (RDP) $(\text{Aff}(\mathcal{S}(E)), 1)$ such that $\tilde{s}(f) := f(s)$, $f \in \text{Aff}(\mathcal{S}(E))$. Then \tilde{s} is a state on $(\text{Aff}(\mathcal{S}(E)), 1)$.

By [Goo, Thm 7.1], the mapping $s \mapsto \tilde{s}$ is an affine homeomorphism between $\mathcal{S}(E)$ and $\mathcal{S}(\text{Aff}(\mathcal{S}(E)), 1)$. \square

Theorem 5.5. *Let E be a pseudo effect algebra with (RDP) having at least one state. Let s be a state on E . Then there is a unique maximal regular Borel probability measure $\mu_s \sim \delta_s$ on $\mathcal{B}(\mathcal{S}(E))$ such that*

$$s(a) = \int_{\mathcal{S}(E)} \hat{a}(x) d\mu_s(x), \quad a \in E. \quad (5.1)$$

Proof. Due to Theorem 5.1, $\mathcal{S}(E)$ is a nonempty Choquet simplex. By Theorem 5.4, there is a unique state \tilde{s} on $(\text{Aff}(\mathcal{S}(E)), 1)$ such that $\tilde{s}(\hat{a}) = s(a)$, $a \in A$.

Applying the Choquet–Meyer Theorem, [Phe, Thm p. 66], we have

$$f(s) = \int_{\mathcal{S}(E)} f(x) d\mu_s, \quad f \in \text{Aff}(\mathcal{S}(E)).$$

Since $\hat{a} \in \text{Aff}(\mathcal{S}(E))$ for any $a \in E$, we have the representation given by (5.1). \square

Theorem 5.6. *Let E be a pseudo effect algebra with (BSP) and let s be a state on E . Then there is a unique regular Borel probability measure, μ_s , on $\mathcal{B}(\mathcal{S}(E))$ such that $\mu_s(\partial_e \mathcal{S}(E)) = 1$ and*

$$s(a) = \int_{\partial_e \mathcal{S}(E)} \hat{a}(x) d\mu_s(x), \quad a \in E. \quad (5.2)$$

Proof. Due to Theorem 5.5, we have a unique regular Borel probability measure $\mu_s \sim \delta_s$ such that (5.1) holds. The characterization of Bauer simplices, [Alf, Thm II.4.1], says that then μ_s is a unique regular Borel probability measure μ_s on $\mathcal{B}(\mathcal{S}(E))$ such that (4.1) holds and $\mu_s(\partial_e \mathcal{S}(E)) = 1$. Hence, (5.2) holds. \square

It is worthy to remark a note concerning formula (5.2) that if μ is any regular Borel probability measure, the right-hand side of formula (5.1) defines a state, say s_μ , on E . But if $\mu(\partial_e \mathcal{S}(E)) < 1$, then for s_μ there is another regular Borel probability measure μ_0 such that $\mu_0(\partial_e \mathcal{S}(E)) = 1$ and it represents s_μ via (5.2).

For example, take E from Example 5.3. The state space of $\mathcal{S}(E)$ is affinely homeomorphic with the real interval $[0, 1]$. Let μ_L be the Lebesgue measure on $[0, 1]$. Formula (5.1) for μ_L defines a state s_L on E such that if $a = (g, h) \in E$ and $s_\lambda(g, h) = \lambda g + (1 - \lambda)h$, then $s_L(a) = \int_0^1 (\lambda g + (1 - \lambda)h) d\mu_L(\lambda) = (g + h)/2$. So that $s_L = (s_0 + s_1)/2$, but $\mu_L(\partial_e \mathcal{S}(E)) = 0$. In other words, s_L has two different representations by regular Borel probability measures via (5.1) (μ_L and $(\delta_0 + \delta_1)/2$, only second one is described by Theorem 5.5) and uniquely via (5.2).

Corollary 5.7. *Let E be a pseudo effect algebra with (RDP) having at least one state. Let m be a Jordan signed measure on E and let $m = \alpha s_1 - \beta s_2$ be its canonical Jordan decomposition. Then there are unique maximal regular Borel probability measures $\mu_{s_1} \sim \delta_{s_1}$ and $\mu_{s_2} \sim \delta_{s_2}$ on $\mathcal{B}(\mathcal{S}(E))$ such that for $\mu_m := \alpha_1 \mu_{s_1} - \alpha_2 \mu_{s_2}$ we have*

$$m(a) = \alpha_1 \int_{\mathcal{S}(E)} \hat{a}(x) d\mu_{s_1}(x) - \alpha_2 \int_{\mathcal{S}(E)} \hat{a}(x) d\mu_{s_2}(x) = \int_{\mathcal{S}(E)} \hat{a}(x) d\mu_m(x)$$

for each $a \in E$.

Proof. Since $m(a) = \alpha_1 s_1(a) - \alpha_2 s_2(a)$, the statement follows from (4.2) and Theorem 5.5. \square

Theorem 5.8. *Let E be a pseudo effect algebra with (BSP) such that E has at least one state. Let m be a Jordan signed measure on E and let $m = \alpha_1 s_1 - \alpha_2 s_2$ be its canonical Jordan decomposition.*

Then there are unique regular Borel probability measures μ_{s_1}, μ_{s_2} on $\mathcal{B}(\mathcal{S}(E))$ such that $\mu_{s_i}(\partial_e \mathcal{S}(E)) = 1$ for $i = 1, 2$ and for $\mu_m := \alpha_1 \mu_{s_1} - \alpha_2 \mu_{s_2}$ we have

$$m(a) = \alpha_1 \int_{\partial_e \mathcal{S}(E)} \hat{a}(x) d\mu_{s_1}(x) - \alpha_2 \int_{\partial_e \mathcal{S}(E)} \hat{a}(x) d\mu_{s_2}(x) = \int_{\partial_e \mathcal{S}(E)} \hat{a}(x) d\mu_m(x)$$

for each $a \in E$.

Proof. It follows from (4.2) and Theorem 5.6. \square

6. CONCLUSION

We have extended the study of representing states on effect algebras by integrals that was started in the paper [Dvu2] for states on pseudo effect algebras, quantum structures where the partial addition is not more assumed to be commutative.

Our research is based on methods of simplices and their application to state spaces. We have showed that every pseudo effect algebra that satisfies the same kind of the Riesz Decomposition Property, (RDP), is always a Choquet simplex, Theorem 5.1. This Theorem extends the result known for effect algebras with (RDP), see [Dvu1, Thm 5.1], for pseudo effect algebras with a stronger version, $(\text{RDP})_1$, that is always an interval in a unital po-group with $(\text{RDP})_1$, and for interval pseudo effect algebras with (RDP), see [Dvu3, Thm 4.3]. We note that we do not know whether every pseudo effect algebra with (RDP) is an interval in a unital po-group.

Finally, this result was applied to represent states on pseudo effect algebras with (RDP) by integrals through regular Borel probability measures, Theorem 5.5 and Theorem 5.6.

It is important to make a finale remark that formulas (5.1) and (5.2) show that they are a bridge between the approach by de Finetti who was a propagator of probabilities as finitely additive measures, and the approach by Kolmogorov for whom a probability measure was a σ -additive measure, [Kol]. The mentioned formulas say by a way that these two approaches are equivalent.

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